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M.Sc. (Sem.-III) Examination October - 2025

Mathematics : Advanced Integral Transforms-I (PGMTH :3043)

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions

(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the above details on your answer book

(2) Follow usual notations and conventions

(3) Figures to the right indicate marks

(4) Answer All Questions

Seat No.:

Student's Signature

Q.1. Attempt Any Two: **14**

(a) If $Z(u_n) = \bar{u}(z)$, then prove that $Z(a^{-n}u_n) = \bar{u}(az)$ and $Z(a^n u_n) = \bar{u}\left(\frac{z}{a}\right)$.

Find Z-transform of $u_n = \dfrac{a^n}{n!}$; $n \geq 0$.

(b) If $\bar{u}(z) = \dfrac{2z^2 + 3z + 4}{(z-3)^3}$, then find u_0, u_1, u_2, u_3.

(c) (i) Find Z-transform of $\cos(n+1)\theta$ and $\sin(n+1)\theta$.

(ii) Evaluate $Z\left(\dfrac{1}{(n+1)!}\right)$ by using shifting theorem and hence evaluate $Z\left(\dfrac{1}{(n+2)!}\right)$

Q.2. Attempt Any Two: **14**

(a) If $Z^{-1}\left[\bar{u}(z)\right] = u_n$ and $Z^{-1}\left[\bar{v}(z)\right] = v_n$, then prove that

$Z^{-1}\left[\bar{u}(z).\bar{v}(z)\right] = \sum_{m=0}^{n} u_m . v_{n-m} = u_n * v_n$.

Also evaluate $Z^{-1}\left[\dfrac{z^2}{(z-2)^2}\right]$ using convolution theorem.

(b) Let $Z(u_n) = \bar{u}(z)$. Then prove that

(i) $u_0 = \lim_{z\to\infty} \bar{u}(z)$

(ii) $\lim_{n\to\infty} u_n = \lim_{z\to 1}(z-1)\bar{u}(z)$

Find the Z-transform of a^{n+3}.

(c) Solve using Z-transform $y_{n+2} + 2y_{n+1} + y_n = n$, given that $y_0 = 0, y_1 = 1$.

Q.3. Attempt Any Two: **14**

(a) Using Z-transform, solve the system of difference equations

$$x_{n+1} - y_n = 1$$
$$y_{n+1} - x_n = 1$$

given $x_0 = 0, y_0 = -1$.

(b) If $M[f(x)] = \overline{f}(p)$, then prove that

(i) $M\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] = \overline{f}(1-p)$

(ii) $M[\log(x) f(x)] = \frac{d}{dp}\overline{f}(p)$

(c) Find the inverse Z-transform of $\dfrac{z^2 - 2z}{(z-2)^3 (z-4)}$

Q.4. Attempt Any Two: **14**

(a) If $M[f(x)] = \overline{f}(p)$, then prove that

(i) $M[f(x^a)] = a^{-1}\overline{f}\left(\frac{p}{a}\right), a > 0$

(ii) $M[f(ax)] = a^{-p}\overline{f}(p)$

(b) If $M[f(x)] = \overline{f}(p)$ and $M[g(x)] = \overline{g}(p)$, then prove that

(i) $M[f(x) * g(x)] = M\left[\int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] = \overline{f}(p)\,\overline{g}(p)$

(ii) $M[f(x) o g(x)] = M\left[\int_0^\infty f(x\xi) g(\xi)\, d\xi\right] = \overline{f}(p)\,\overline{g}(1-p)$

(c) Define Mellin transform. Find Mellin transform of $f(x) = \frac{1}{1+x}$.

Q.5. Attempt Any Two: **14**

(a) Using Mellin transform, solve the boundary value problem

$x^2 u_{xx} + x u_x + u_{yy} = 0, 0 \le x < \infty, 0 < y < \infty$

$u(x,0) = 0$ and $u(x,1) = \begin{cases} A; 0 \le x \le 1 \\ 0; x > 1 \end{cases}$

(b) Use Mellin transform to solve the following integral equations

(i) $\int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} = h(x)$

(ii) $\int_0^\infty f(\xi) k(x,\xi) d\xi = g(x), x > 0$

(c) Using Mellin transform, find Potential $\boldsymbol{Q}(r,\theta)$ that satisfies the Laplace Equation $r^2\boldsymbol{Q}_{rr} + r\boldsymbol{Q}_r + \boldsymbol{Q}_{\theta\theta} = 0$ in an infinite wedge $0 < r < \infty, -\alpha < \theta < \alpha$ with the boundary conditions

$\boldsymbol{Q}(r, \alpha) = f(r), \boldsymbol{Q}(r, -\alpha) = g(r), 0 < r < \infty$

$\boldsymbol{Q}(r,\theta) \to 0$ as $r \to \infty, \forall \theta$ in $-\alpha < \theta < \alpha$